Optical realization of a quantum beam splitter

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We show how the quantum process of splitting light may be modelled in classical optics. A second result is the possibility to engineer specific forms of a classical field.

A. Introduction

The modelling of quantum mechanical systems with classical optics is a topic that has attracted interest recently. Along these lines Man'ko *et al.* have proposed to realize quantum computation by quantum like systems [1] and Crasser *et al.* [2] have pointed out the similarities between quantum mechanics and Fresnel optics in phase space. Following these cross-applications, here we would like to show how a quantum beam splitter may be modelled in classical optics. The possibility of generating specific forms (engineering) of the propagated field is also studied.

A beam splitter is an optical component that combines two propagating modes into two other propagating modes. Fig. 1 shows the setup that produces this effect for a 50:50 beam splitter. Two fields a_x and a_y enter the beam splitter and two fields exit it, that correspond to a combination of the entering ones. In quantum optics the beam splitter is modelled by the interaction of two fields [3] with the Hamiltonian given by

$$H = \omega_x a_x^\dagger a_x + \omega_y a_y^\dagger a_y + \chi(a_y^\dagger a_x + a_x^\dagger a_y) \tag{1}$$

where the ω's are the field frequencies, χ is the interaction constant and a_j and $a_j^\dagger$ $(j = x, y)$ are the annihilation and creation operators of the field modes.

If we consider equal field frequencies, we can obtain the beam splitter operator [4]

$$B = e^{-i\theta(a_y^\dagger a_x + a_x^\dagger a_y)} \tag{2}$$

with $\theta = \chi t$. Note that

$$\begin{aligned} B a_x B^\dagger &= \cos(\theta) a_x + i\sin(\theta) a_y, \\ B a_y B^\dagger &= \cos(\theta) a_y + i\sin(\theta) a_x \end{aligned} \tag{3}$$

such that $\cos(\theta)$ and $\sin(\theta)$ may be related directly with the transmission and reflection coefficients. One feature present in the quantum beam splitter is that, if there is only one field entering by one of the arms of the beam splitter, one has to consider always a vacuum field entering by the other arm. It is well known that this system produces entanglement [5, 6], for instance, if we consider the 50:50 beam splitter, i.e. we set $\theta = \pi/4$, (see Fig. 1) and in each of the arms the first excited number state, namely the state $|\psi_I\rangle = |1\rangle_x |1\rangle_y$ as initial state, we have as final state

$$|\psi_F\rangle = \frac{i}{\sqrt{2}}(|2\rangle_x|0\rangle_y + |0\rangle_x|2\rangle_y), \tag{4}$$

this is an entangled state that tells that both photons travel together.

B. Modelling field-field interaction

We consider the paraxial propagation of a field, that has the equation form (see for instance [7]

$$2ik_0 \frac{\partial E}{\partial z} = \nabla_\perp^2 E + k^2(x, y)E \tag{5}$$

where $k_0 = 2\pi n_0/\lambda$ is the wavenumber with λ the wavelength of the propagating lightbeam, n_0 is the homogeneous refractive index. The function $k^2(x, y)$ describes the inhomogeneity of a medium responsible for the waveguiding of an optical field E. It has been recently shown that by using an astigmatic and slightly tilted probe beam, it produces a GRIN like medium, with an extra cross term χxy [7]. Taking into account such inhomogeneity the paraxial wave equation is written as

$$\begin{aligned} i\frac{\partial E}{\partial z} &= \frac{\nabla_\perp^2}{2k_0} E \\ &+ \left(k_0/2 - \frac{1}{2}(k_0\tilde{\alpha}_x^2 x^2 + k_0\tilde{\alpha}_y^2 y^2) + \chi xy\right) E, \end{aligned} \tag{6}$$

with $k_0\tilde{\alpha}_q^2, \qquad q=x,y$ inhomogeneity parameters related to the generated GRIN medium [7].

C. Optical realization of the quantum beam splitter

Let us define the *ladder* operators [8]

$$a_q = \sqrt{\frac{k_0\tilde{\alpha}_q}{2}}q + \frac{1}{\sqrt{2k_0\tilde{\alpha}_q}}\frac{d}{dq},$$
$$a_q^\dagger = \sqrt{\frac{k_0\tilde{\alpha}_q}{2}}q - \frac{1}{\sqrt{2k_0\tilde{\alpha}_q}}\frac{d}{dq}, \tag{7}$$

with $q=x,y$. These operators are also called creation and annihilation operators in quantum optics because the action of a_x on a function

$$u_m(x) = \left(\frac{k_0\tilde{\alpha}_x}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^m m!}}H_m(\sqrt{k_0\tilde{\alpha}_x}x)e^{-k_0\tilde{\alpha}_x x^2/2} \tag{8}$$

where $H_m(x)$ are Hermite polynomials, gives

$$a_x u_m(x) = \sqrt{m}u_{m-1}(x) \tag{9}$$

and

$$a_x^\dagger u_m(x) = \sqrt{m+1}u_{m+1}(x). \tag{10}$$

The above equations are also valid for the y coordinate simply making the change $y \to x$. Furthermore, note that $[a_x, a_x^\dagger]=1$. We can rewrite (6) using (7) as

$$i\frac{\partial E}{\partial z} = \left(\tilde{\alpha}_x(n_x+\frac{1}{2}) + \tilde{\alpha}_y(n_y+\frac{1}{2}) + \chi xy + \frac{k_0}{2}\right)E, \tag{11}$$

with $n_j = a_j^\dagger a_j$ for $j=x,y$ the so-called number operator. We transform (11) to get rid of the constant terms, via $\psi = \exp[-iz(\tilde{\alpha}_x+\tilde{\alpha}_y+k_0)/2]E$, and we assume $\chi \ll \tilde{\alpha}_x, \tilde{\alpha}_y$ so we can perform the so-called rotating wave approximation (see for instance [9]) to finally obtain

$$i\frac{\partial \psi}{\partial z} = \left(\tilde{\alpha}_x n_x + \tilde{\alpha}_y n_y + \tilde{\chi}(a_x a_y^\dagger + a_x^\dagger a_y)\right)\psi, \tag{12}$$

with $\tilde{\chi} = \frac{\chi}{2k_0\sqrt{\tilde{\alpha}_x\tilde{\alpha}_y}}$. This equation is equivalent to the field-field interaction in quantum optics [10].

We do a last transformation $\phi = \exp[-iz\tilde{\alpha}_x(n_x+n_y)]\psi$ and obtain

$$i\frac{\partial \phi}{\partial z} = [\Delta n_y + \tilde{\chi}(a_x a_y^\dagger + a_x^\dagger a_y)]\phi \equiv H\phi, \tag{13}$$

with $\Delta = \tilde{\alpha}_y - \tilde{\alpha}_x$. It is useful to define "normal-mode" operators by [10]

$$A_1 = \delta a_x + \gamma a_y, \qquad A_2 = \gamma a_x - \delta a_y, \tag{14}$$

with

$$\delta = \frac{2\tilde{\chi}}{\sqrt{2\Omega(\Omega-\Delta)}}, \qquad \gamma = \sqrt{\frac{\Omega-\Delta}{2\Omega}}, \qquad \delta^2+\gamma^2=1 \tag{15}$$

with $\Omega = \sqrt{\Delta^2+4\tilde{\chi}^2}$ the Rabi frequency. A_1 and A_2 are annihilation operators just like a and b and obey the commutation relations

$$[A_1, {A_1}^\dagger] = [A_2, A_2^\dagger] = 1, \tag{16}$$

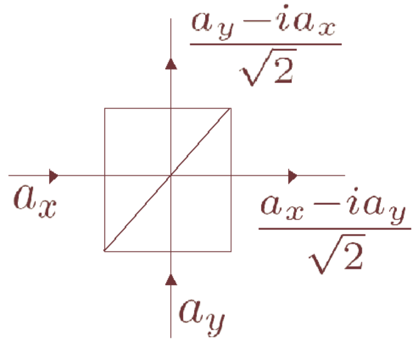

FIG. 1: Configuration of the beam splitter (50:50) operation.

moreover, the normal-mode operators commute with each other

$$[A_1, A_2] = [A_1, A_2^\dagger] = 0. \tag{17}$$

In terms of these operator H in (13) becomes

$$H = \mu_1 A_1^\dagger A_1 + \mu_2 A_2^\dagger A_2, \tag{18}$$

with $\mu_{1,2} = (\Delta \pm \Omega)/2$. In order to have a way of transforming functions from one basis to the other, we note that the lowest functions $u_0(x)u_0(y)$ are also eigenfunctions of the normal-mode operators

$$A_m u_0(x)u_0(y) = 0, \qquad m = 1, 2, \tag{19}$$

i.e. they are the lowest states, up to a phase, in the new basis [10]

$$u_0(x)u_0(y) = U_0^1(x,y)U_0^2(x,y). \tag{20}$$

1. *Initial function $u_1(x)u_1(y)$*

If we consider

$$\phi(z=0; x, y) = u_1(x)u_1(y) \tag{21}$$

the propagated function reads

$$\phi(z; x, y) = e^{-iz(\mu_1 A_1^\dagger A_1 + \mu_2 A_2^\dagger A_2)} a_x^\dagger a_y^\dagger u_0(x)u_0(y), \tag{22}$$

by writing the creation operators in terms of creation operators for the normal-modes, (14), we obtain

$$\phi(z; x, y) = e^{-iz\mu_1 A_1^\dagger A_1} e^{-iz\mu_2 A_2^\dagger A_2} (\delta A_1^\dagger + \gamma A_2^\dagger)(\gamma A_1^\dagger - \delta A_2^\dagger) U_0^1(x,y)U_0^2(x,y). \tag{23}$$

Now we use the properties $e^{-iz\mu_j A_j^\dagger A_j} A_j^\dagger e^{iz\mu_j A_j^\dagger A_j} = A_j^\dagger e^{-iz\mu_j}$ and $e^{-iz\mu_j A_j^\dagger A_j} U_0^j(x,y) = U_0^j(x,y)$ to obtain

$$\phi(z;x,y) = (\delta A_1^\dagger e^{-iz\mu_1} + \gamma A_2^\dagger e^{-iz\mu_2})(\gamma A_1^\dagger e^{-iz\mu_1} - \delta A_2^\dagger e^{-iz\mu_2}) U_0^1(x,y) U_0^2(x,y), \tag{24}$$

that in the old basis reads

$$\phi(z;x,y) = [\eta(z)\beta(z) a_x^{\dagger 2} + \epsilon(z)\beta(z) a_y^{\dagger 2} + (\beta^2(z) + \epsilon(z)\eta(z)) a_x^\dagger a_y^\dagger] u_0(x) u_0(y), \tag{25}$$

with $\eta(z) = \delta^2 e^{-iz\mu_1} + \gamma^2 e^{-iz\mu_2}$, $\beta(z) = \gamma\delta(e^{-iz\mu_1} - e^{-iz\mu_2})$ and $\epsilon(z) = \delta^2 e^{-iz\mu_2} + \gamma^2 e^{-iz\mu_1}$. The term that multiplies $a_x^\dagger a_y^\dagger$ may be written as

$$e^{-i\Delta z}[(\gamma^2 - \delta^2)^2 + 4\gamma^2\delta^2 \cos(\Omega z)],$$

oscillates crossing zero periodically. By looking at the propagated field at one of these zeros (at z_0), a state of the form (4) is obtained, i.e.

$$\phi(z_0;x,y) = \eta(z_0)\beta(z_0) u_2(x) u_0(y) + \epsilon(z_0)\beta(z_0) u_0(x) u_2(y). \tag{26}$$

2. SU(2) coherent state

One can engineer functions in the form of determined superpositions of the functions (8). For instance, one can engineer the so-called SU(2) coherent state by setting the functions $u_m(x)$ and $u_0(y)$ at the plane $z = 0$

$$\phi(z=0;x,y) = u_m(x) u_0(y) = \frac{a_x^{\dagger m}}{\sqrt{m!}} u_0(x) u_0(y), \tag{27}$$

that after application of the propagator $e^{-iz(\mu_1 A_1^\dagger A_1 + \mu_2 A_2^\dagger A_2)}$ gives

$$\phi(z;x,y) = \frac{(\eta(z) a_x^\dagger + \beta(z) a_y^\dagger)^m}{\sqrt{m!}} u_0(x) u_0(y), \tag{28}$$

that may be rewritten as the SU(2) coherent state [11, 12]

$$\phi(z;x,y) = \sum_{n=0}^{m} \begin{pmatrix} m \\ n \end{pmatrix}^{1/2} \eta^{m-n}(z)^n \beta(z) u_{m-n}(x) u_n(y). \tag{29}$$

3. Gaussian function

Consider now that at $z = 0$ we have a displaced Gaussian function as a function of x and a function $u_0(y)$,

$$\phi(z=0;x,y) = \left(\frac{k_0\tilde{\alpha}_x}{\pi}\right)^{1/4} e^{-(\alpha - x\sqrt{\frac{k_0\tilde{\alpha}_x}{2}})^2} u_0(y) \equiv e^{-\frac{\alpha^2}{2}} \sum_n^{\infty} \frac{\alpha^n}{\sqrt{n!}} u_n(x) u_0(y), \tag{30}$$

by using the Glauber displacement operator[13], this function may be rewritten as

$$\phi(z=0;x,y) = D_{a_x}(\alpha) u_0(x) u_0(y), \tag{31}$$

with $D_{a_x}(\alpha) = \exp[\alpha(a_x^\dagger - a_x)]$. After application of the propagator operator, the propagated field reads

$$\phi(z;x,y) = D_{a_x}[\alpha\eta(z)] u_0(x) D_{a_y}[\alpha\beta(z)] u_0(y). \tag{32}$$

The electromagnetic field then will be displaced Gaussians in x and y dimensions, the initial Gaussians at $z = 0$ will exchange displacement, i.e. the displaced Gaussian (that depends on x) will periodically diminish its displacement and back, while the one centered at $y = 0$ will periodically be displaced and back to the origin.

D. Conclusions

We have shown how to model a quantum beam splitter by propagating electromagnetic fields. Specific electromagnetic fields may be engineered by using this modelling, in particular we have shown how SU(2) coherent functions may be realized.

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